Optimal Trading for Mean-Reverting Security in Finite Time with Transaction Fees

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A Thesis
In
The John Molson School of Business

Presented in Partial Fulfillment of the Requirements
For the Degree of Master of Science in Administration(FINANCE)
Concordia University
Montréal, Québec, Canada

Apr 2016

Concordia University

School of Graduate Studies

This is to certify that the thesis prepared

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Entitled: **Optimal Trading for Mean-Reverting Security in Finite Time with Transaction Fees**

and submitted in partial fulfillment of the requirements for the degree of

Master of Science in Administration(Finance)

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ABSTRACT

Optimal Trading for Mean-Reverting Security in Finite Time with Transaction Fees

Chang Xiao

The optimal trading strategy of a mean-reverting security, which follows the Ornstein-Uhlenbeck process, is considered for investors facing the fixed transaction fee and the proportional transaction fee, which is proportional to the number of trading shares, and trading in finite time. The mean-reverting feature is applied in deriving partial differential equations with optimal trading boundaries from the value function. The optimal trading boundaries include optimal trading prices, optimal positions after trading. Analytical solutions for optimal trading problems are obtained by theoretical analysis of partial differential equations and the optimal trading strategy is obtained by computational analysis for the optimal boundary conditions. The optimal trading strategy includes several optimal trading prices and optimal positions.

ACKNOWLEDGMENTS

In my way of pursuing this M.Sc. degree in the past two years, so many people contributed a lot, in many different ways, to make my success as a part of their own.

I owe many thanks to my supervisor Dr. Isaenko, who does not just share his knowledge and experience in finance research, but also encourage me in research. I owe many thanks to professors who have taught me finance courses. They are Dr. Shanker, Dr. Mateti, Dr. Ravi, Dr. Switzer, Dr. Newton, Dr. Ullah, Dr. Tapiero, Dr. Basu and Dr. Paeglis. I also owe many thanks to our staff Sandra and Edite, who work hard to make all graduate students can focus on the research. Finally, I have to thank my wife Felissa, and my parents. Without their love and support, I could not insist in challenging tough problems.

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CHAPTER 1: Introduction

1.1 Introduction to the mean reversion security price

Trading mean-reverting security has been very popular among investors for years. The key feature of such security is that it predictably fluctuates around its mean value. One of the most common process that has a mean-reverting property is the Ornstein-Uhlenbeck process(OU) given below:

$$dS = \kappa(\bar{S} - S)dt + \sigma_S dw. \tag{1.1}$$

The security price curve in the second figure of Fig 1.1 is numerically generated by OU process with the security price S starting at 35 and fluctuating around the mean price 40 with the mean-reverting rate 0.1 and the volatility 0.5. The third figure of Fig 1.1 shows that the returns of the generated security price follow normal distribution.

It follows from the equation above that if the price is more than $\bar{S}$, then the drift of the price is negative and the price tends to decrease, while it increases if the price is less than $\bar{S}$. Therefore, if investors trade T-bill and a mean-reverting portfolio, then they should short stock when it is expensive since the stock price is expected to come down and investors buy when the stock price is low since it is expected to increase. The detailed trading strategy can be found in [16]. Unfortunately, trading a mean-reverting portfolio in practice is marked by the presence of bid-ask spread and fixed transaction fees. The losses on the spread and fees can accumulate to a very large amount over time especially if trading is done frequently. Therefore, it is important for an investor to find one optimal trading strategy when trading a mean-reverting security

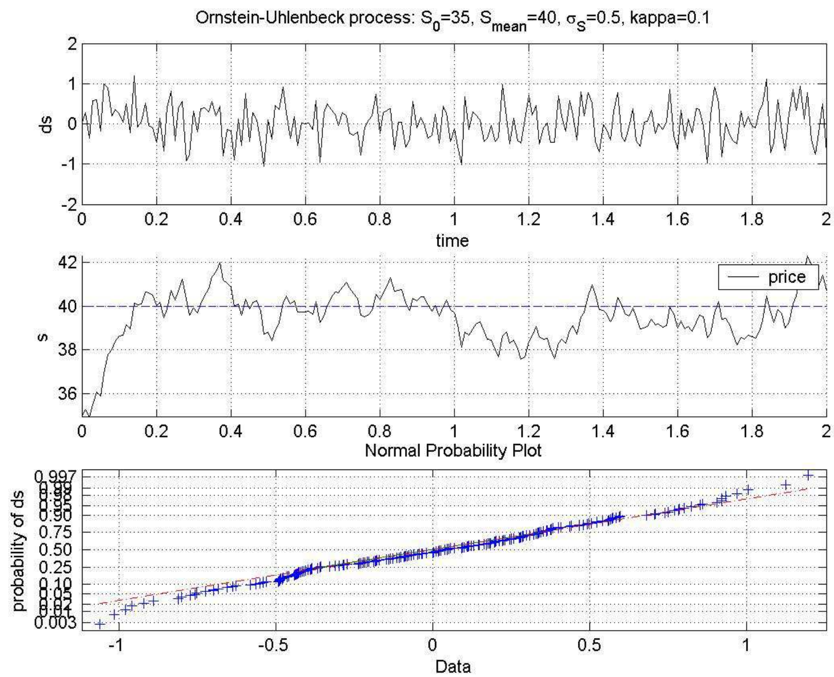


Figure 1.1: Security price following Ornstein-Uhlenbeck process

involves paying fixed transaction fees and facing a bid-ask spread, which is represented by the proportional transaction fees.

1.2 Literature review

The optimal trading with mean-reverting security has been studied in [16], in which the utility function is the sum of discounted portfolio value. The optimal trading without transaction cost is given in [9]. The optimal trading for infinite time horizon with transaction costs is presented in [7, 11]. The optimal trading for finite horizon with transaction costs is shown in [1, 8]. The optimal investment with transaction fees and capital gains taxes is studied in [5].

The evidence of mean reversion in security price is implied in [13] by autocorrelation analysis over different time horizons. The optimal trading for mean-reverting price spreads is studied in [6] by considering transaction costs and stop-loss exit for risk control.

From the point of view of dynamic system, OU process is a stochastic process with one attractor. A low dimensional attractor observed in S&P500 with a positive Lyapunov exponent, which indicates its chaotic feature, is presented in [12].

1.3 Structure of this thesis

In this thesis, the optimal trading problems for one mean-reverting security is studied. The optimal trading strategies for multiple securities can be developed from the result for one security. The Value function for optimal trading problem in finite time is derived in section 2.1. The optimal number of shares for the case without transaction fee is solved and the simple proof of uniqueness is given in section 2.2. For the optimal trading in finite time with proportional transaction fee only, basic models with an analytical solution are given in section 3.1, the optimal trading strategy are derived numerically in section 3.2 and a simple proof for the uniqueness of the analytical solution is given in section 3.3. The basic models in section 4.1 for the cases with both fixed transaction fee and proportional transaction fee are different from the cases with only proportional transaction fee. Then analytical solutions and numerical optimal trading strategy are developed in section 4.2 and section 4.3 for the cases with both fees.

CHAPTER 2: Optimal Trading in Finite Time without Transaction Fees

2.1 Derivation of value function

We assume that the security price S follows OU process as described in (1.1) and the risk-free rate is zero. We consider the investors' utility function as

$$U = E_0(-\frac{1}{\gamma}\exp(-\gamma X_T)), \tag{2.1}$$

in which there are two parameters, γ is the coefficient of absolute risk aversion, and X_T is the discounted value of the total portfolio at finite time T.

As there is a constraint (1.1) on security price, the models for optimal trading specified by the number of shares N, unlike models in [7, 8]. Let D and I represent the cumulative number of shares in short position and long position. And let X represent the total value of portfolio. Then the equations for trading process with transaction fees are

$$dX = N\mu_S dt + N\sigma_S dW - \delta S dI - \alpha S dD + F 1_{dI+dD>0} \tag{2.2}$$

$$dN = dI - dD \tag{2.3}$$

where δ, α are the rates of proportional transaction fee for trading volume, and F is the fixed transaction fee for each transaction. Thereafter, investors are facing the optimization problem with the value function $V(X, S, N, t)$ at time t

$$V(X, S, N, t) = \sup_{(I,D)} E_t[(-\frac{1}{\gamma}\exp(-\gamma X_T))|X(t) = X, S(t) = S, N(t) = N], \tag{2.4}$$

and this value function solves the Bellman equation

$$0 = \max_{N\in R}[V_t + \frac{1}{2}N^2\sigma_S^2 V_{XX} + \frac{1}{2}\sigma_S^2 V_{SS} + N\sigma_S^2 V_{XS} + \mu_S N V_X + \mu_S V_S], \tag{2.5}$$

where σ_S and $\mu_S = \kappa(\bar{S} - S)$ are terms from OU process as in (1.1). To solve this optimal trading problem, we need to make conjectures on the patterns of the value function. The case without transaction cost is slightly different from cases with transaction cost, as the case without transaction cost is independent of the number of share.

2.2 Model and unique solution for the case without transaction fee

Considering optimal trading without transaction fees, such as $\alpha = 0$ and $\beta = 0$, as the first order partial derivative of (2.5) with respect to N is equal to zero in the case without transaction fees, we have

$$N^* = -\frac{\mu_S V_X + \sigma_S^2 V_{XS}}{\sigma_S^2 V_{XX}}. \tag{2.6}$$

Also we may conjecture the pattern of value function is

$$V(X, S, t) = -\frac{1}{\gamma}\exp(-\gamma X + g(S, t)), \tag{2.7}$$

which is independent of the number of shares for the transaction fee is always zero in this case. Inserting (2.6) and (2.7) into (2.5) to eliminate N, then we have the equations for g(S,t)

$$0 = g_t - \frac{\mu_S^2}{2\sigma_S^2} + \frac{\sigma_S^2}{2} g_{SS} \tag{2.8}$$

$$g(S, T) = 0, \tag{2.9}$$

where the terminal boundary condition (2.9) indicates that the investor has to clear all positions at the end. The analytical solution for the case without transaction fee is

$$g(S, t) = -\frac{\kappa^2}{2}(T - t)(\frac{S - \overline{S}}{\sigma_S})^2 - (\frac{\kappa(T - t)}{2})^2. \tag{2.10}$$

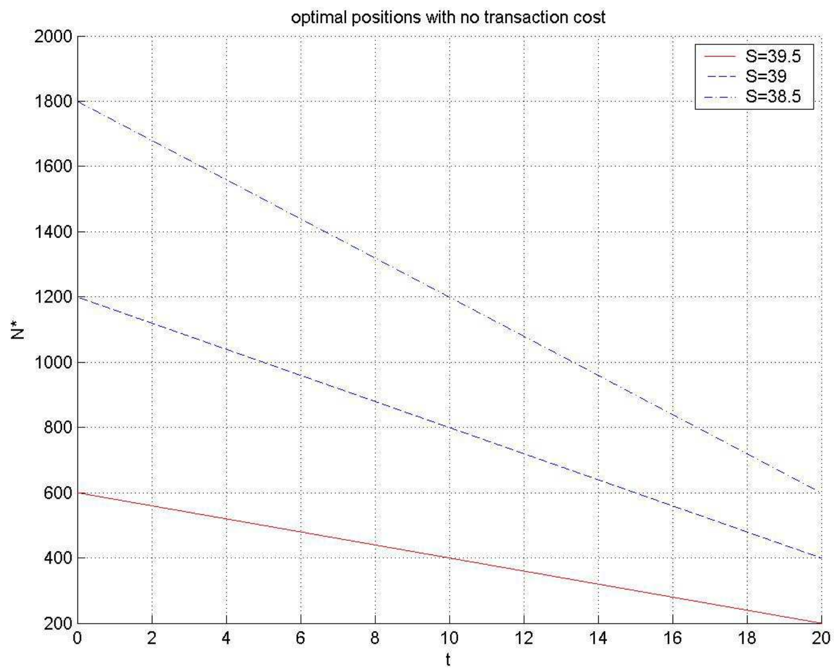


Figure 2.1: Optimal positions in the case without transaction fees

It is not difficult to verify (2.10) is the solution satisfying (2.8-2.9), and (2.8) is in fact a parabolic equation [10]. Applying this solution (2.10) to the value function (2.7) and the optimal number of shares (2.6), we have the optimal strategy in the case without transaction fee as have the position

$$N^*(S,t) = \kappa(\bar{S} - S)\frac{1 + \kappa(T - t)}{\gamma\sigma_S^2},$$

which suggest investors allocate proportionally to the stock deviation from its mean and decrease in magnitude as $t \to T$ in Fig 2.1. And the proof for the uniqueness of (2.10) in C^∞ of this solution is simply introduced as below.

By setting $x = \frac{S-\bar{S}}{\sigma_S}$ to simplify the discussion of uniqueness, we have $g(S,t) = g(\bar{S} + x\sigma_S, t) =$

$q(x,t)$. Then equations (2.8-2.9) for the case without transaction fee become

$$0 = q_t - \frac{\kappa^2 x^2}{2} + \frac{1}{2} q_{xx} \tag{2.11}$$

$$q(x,T) = 0 \tag{2.12}$$

Without losing generality, we may investigate the solutions with $x \in [-L, L]$ and afterwards let $L \to \infty$. With $x \in [-L, L]$, $g(S,t) = q(x,t) \in C^\infty$ and (2.11-2.12) are symmetric with $x = 0$, we can have an analytical solution in the pattern of Fourier series

$$q(x,t) = -\frac{\kappa^2}{2}(T-t)x^2 - (\frac{\kappa(T-t)}{2})^2 + \sum_{k=-\infty}^{\infty} [C_k \exp(\frac{1}{2}(\frac{\Pi}{L})^2 k^2 t) \exp(i * \frac{\Pi}{L} kx)] \tag{2.13}$$

where C_k is constant and C_k has to be zero for satisfying the boundary condition (2.12). Thereafter, let $L \longrightarrow \infty$ in the Fourier series, it is feasible to claim that (2.10) is the unique solution in C^∞ for (2.8-2.9).

CHAPTER 3: Optimal Trading in Finite Time with Proportional Fees

3.1 An analytical solution for the case with proportional fees

In the cases with proportional transaction fees, the fixed cost F equals to zero and the proportional buying and selling fee rates δ, α are positive. Due to positive transaction fees, the trading activity should not be as frequent as the case without transaction fees and the value function for the cases with transaction fees is also depend on the number of shares, unlike (2.7) for the case without transaction fees.

$$V(X, S, N, t) = -\frac{1}{\gamma}\exp(-\gamma X + g(S, N, t)). \tag{3.1}$$

First of all, for finite time problem, the investor has to clear all position at least at time T, then we have a terminate boundary condition as

$$g(S, N, T) = 0 \tag{3.2}$$

As mentioned in[2, 7, 8, 14], there are two optimal trading boundaries, such as optimal buying price $\underline{S}(N, t)$ and optimal selling price $\overline{S}(N, t)$, dividing the whole domain into two parts: no-transaction region(NT) and a transaction region. For convenience, we use ST to indicate the transaction region above the optimal selling boundary and use BT to indicate the transaction region below the optimal buying boundary. The optimal strategy for the investors is to buy enough amount to reach the NT when their position is in BT and to sell enough amount to reach the NT when their position is in ST. When the investors are in NT, their value function follows (2.5). When they are in BT, they have to buy the amount dI in order to reach the NT boundary: $V(X, \underline{S}, N, t) = V(X - \delta\underline{S}dI, \underline{S}, N + dI, t)$. When they are in ST, they have to sell the amount dD in order to reach the NT boundary: $V(X, \overline{S}, N, t) = V(X - \alpha\overline{S}dD, \overline{S}, N - dD, t)$. Considering the conjecture for the value function (3.1), we have

$$g_N(\underline{S}, N, t) = -\gamma\delta\underline{S} \tag{3.3}$$

for the optimal buying boundary and

$$g_N(\overline{S}, N, t) = \gamma\alpha\overline{S} \tag{3.4}$$

for the optimal selling boundary. The spatial boundary conditions (3.3) and (3.4) will conflict with the terminate boundary condition (3.2) at $t = T$ and also investors usually have to stop trading a bit before the terminate time T, therefore the optimal buying and selling boundary is for the time $t < T$. And moreover, by inserting (3.1) into (2.5), we have the basic equation in NT as

$$0 = g_t + \frac{1}{2}(\gamma N\sigma_S)^2 - \gamma\mu_S N + \frac{1}{2}\sigma_S^2(g_S^2 + g_{SS}) + (\mu_S - \gamma N\sigma_S^2)g_S \tag{3.5}$$

Now we convert the problem of optimal trading in finite time with proportional fees into a partial differential equation problem involving equation (3.5) and boundary conditions (3.2),(3.3) and (3.4).

In [7], they made conjectures for the pattern of BT and ST and then use the continuity condition at the optimal buying and selling boundaries to solve the optimal trading problem. However, the mean-reverting security is different. The security price is close to $\bar{S}$ at time T in the mean-reverting case and the equation (3.5) is approaching equations for travelling waves as the mean-reverting strength κ increases and the volatility σ_S decreases. By the feature of characteristic curve in travelling waves and the requirement of one solution for the optimal trading prices $(\underline{S}, \overline{S})$, we build an analytical solution for the case with proportional transaction fees. And the uniqueness can be proved simply by applying the terminate condition (3.2) along the characteristic curves.

$$g(S,N,t) = \frac{\gamma^2 N^2 \sigma_S^2}{4\kappa}(1 - \exp(-2\kappa T + 2\kappa t)) + \gamma N(S - \bar{S})(1 - \exp(-\kappa T + \kappa t)) \tag{3.6}$$

with optimal buying price

$$\underline{S} = \frac{(1 - \exp(-\kappa T + \kappa t))(\bar{S} - \frac{\gamma N \sigma_S^2}{2\kappa}(1 + \exp(-\kappa T + \kappa t)))}{1 - \exp(-\kappa T + \kappa t) + \delta} \tag{3.7}$$

and optimal selling price

$$\overline{S} = \frac{(1 - \exp(-\kappa T + \kappa t))(\bar{S} - \frac{\gamma N \sigma_S^2}{2\kappa}(1 + \exp(-\kappa T + \kappa t)))}{1 - \exp(-\kappa T + \kappa t) - \alpha}. \tag{3.8}$$

The denominator in the formula of optimal selling price will be zero at

$$t = t_c = T + \frac{\ln(1-\alpha)}{\kappa} \tag{3.9}$$

which indicates the trading activity will be terminated before T by $dt_c = T - t_c = -\frac{\ln(1-\alpha)}{\kappa}$ and this gap is independent of finite time T. For the investors facing mean-reverting security with proportional transaction fee, their optimal trading strategy is to buy enough amount to reach NT when security price is less than optimal buying price, and to sell enough amount to reach NT when security price is more than optimal selling price. The amount of trading can be obtained from the inverse functions of $\underline{S}(N,t)$ and $\overline{S}(N,t)$ numerically as shown in the next section.

3.2 Numerical result for the case with proportional fees

The time gap $dt_c = T - t_c$ is shown in Fig 3.1 with $\alpha = \delta = 0.0025$. We use .5 as the time gap before terminate time in our numerical simulations. In fact, only the formula of optimal selling

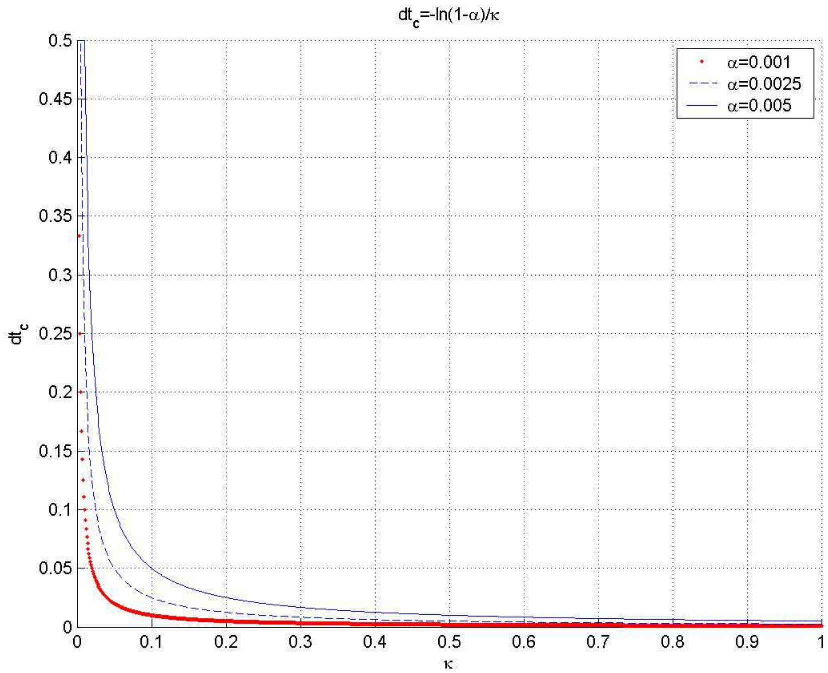


Figure 3.1: Time gap before stopping trading

price (3.8) has a singularity at t_c, while the optimal buying price (3.7) and function (3.6) are always continuous.

Without specification, the parameters in numerical simulation are set to $T = 20, \gamma = 0.001, \sigma_S = 0.5, \bar{S} = 40, \kappa = 0.1, N = 100, \alpha = \delta = 0.0025$ in this thesis. The optimal trading prices at different times are functions depending on the number of shares as in Fig 3.2. The optimal buying price (3.7) and optimal selling prices (3.8) linearly decrease as the number of shares increase, and the rate of decreasing increases along the time as the curves steepen as t getting close to terminate time in Fig 3.2. In the lower two figures in Fig 3.2, the gaps between optimal buying price and optimal selling price increase along time. Thus the investors have to ask for more potential profit to make trading decisions as time increases.

For different time ranges, such as $T = 20, 40, 60$ in Fig 3.3, the optimal trading prices are close

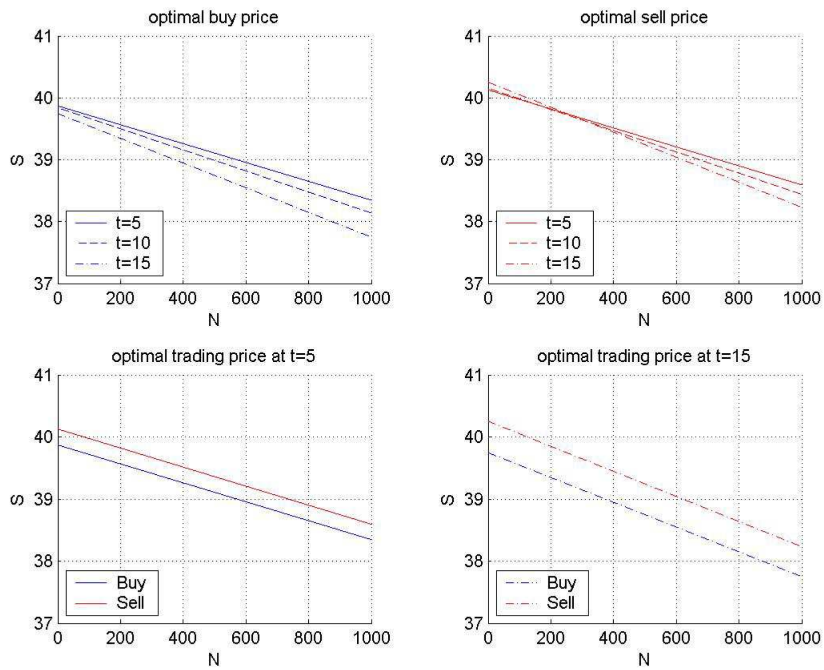


Figure 3.2: Optimal trading prices in finite time with proportional transaction fee at different time points

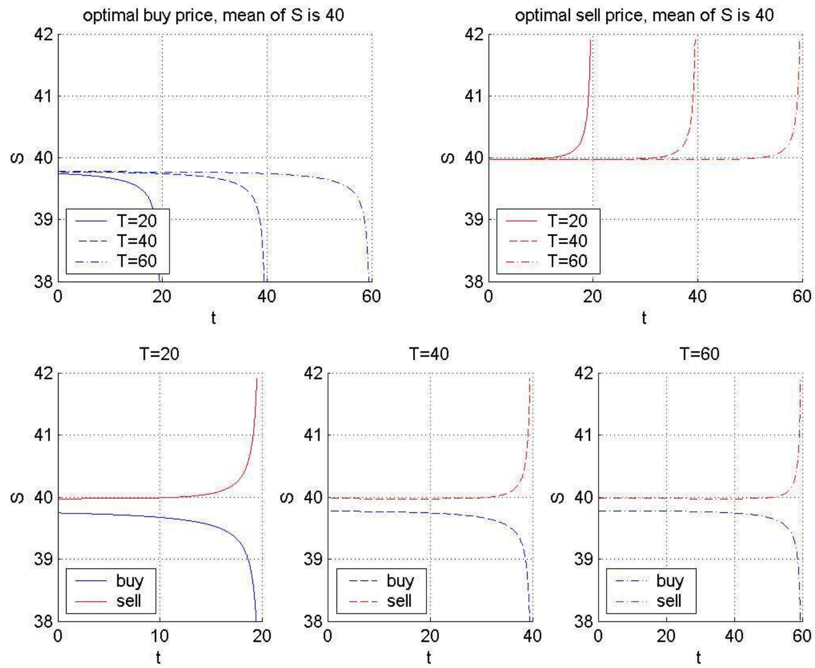


Figure 3.3: Optimal trading in finite time with proportional transaction fee for different time ranges

for the time t far from maximum time T. And investors have to buy at much lower price and sell at much higher price as time approaches the terminate time T.

As the relative positions of curves for optimal trading price in the Fig 3.4, the investors should buy in at lower price and sell at higher price for larger volatility. For example, by Fig 3.4, at the initial time, the investor's optimal buying price is about 39.3 for $\sigma_S = 1$, while his optimal buying price is about 39.9 for $\sigma_S = 0.1$. According to the curves for $\sigma_S = 0.5$ and $t < 15$ in Fig 3.4, the optimal buying price is between 39.5 and 39.75, while the optimal selling price is between 40 and 40.05.

For more proportional transaction fees, the optimal buying price is lower and the optimal selling price is higher as in Fig 3.5. Without transaction fee, the optimal buying price is the

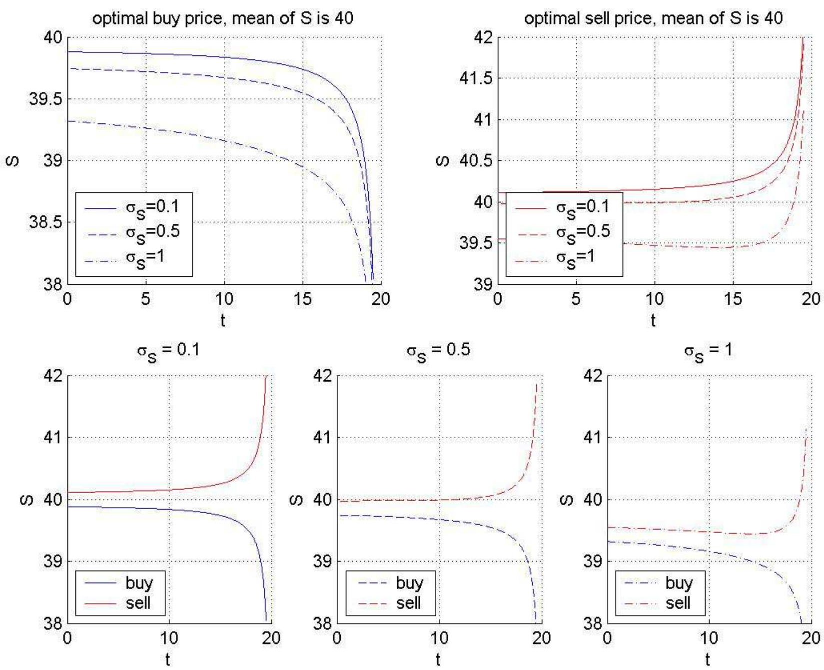


Figure 3.4: Optimal trading in finite time with proportional transaction fee for different volatilities

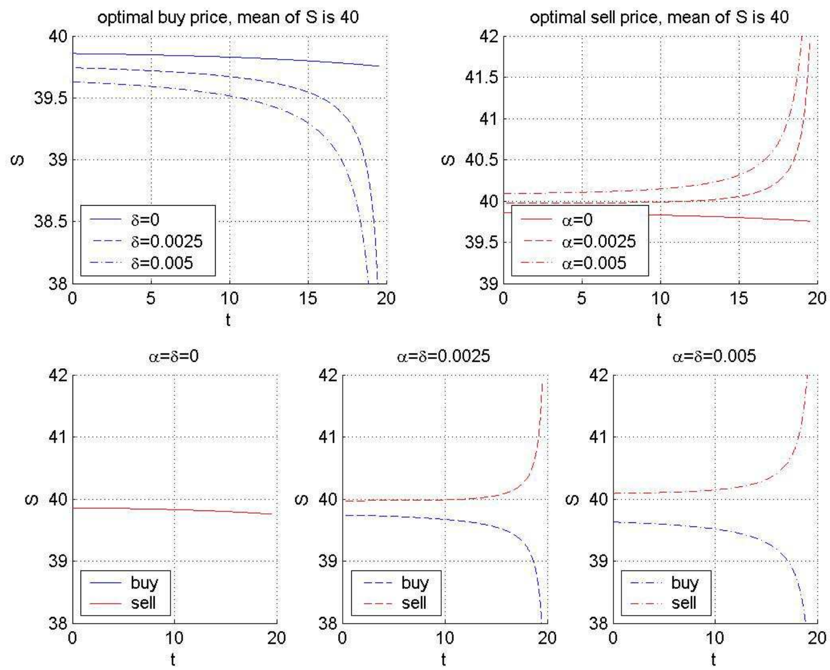


Figure 3.5: Optimal trading in finite time with proportional transaction fee for different proportional transaction fee rates

same as the optimal selling price for the two curves of $\delta = \alpha = 0$ overlap in Fig 3.5. There is only a bit difference between the optimal trading price for $\delta = \alpha = 0$ in Fig 3.5 and the inverse function of (2.1), for there is still a singularity at $t = T$ for $\delta = \alpha = 0$ in (3.7) and (3.8).

3.3 Proof of uniqueness

As we have already built an analytical solution (3.6) for (3.5) with the terminate boundary condition (3.2), we can assume the pattern of any solution is equal to this analytical solution plus φ, which is the solution for the homogeneous version of (3.5):

$$0 = \varphi_t + \frac{1}{2}\sigma_S^2(\varphi_S^2 + \varphi_{SS}) + (\mu_S - \gamma N \sigma_S^2 \exp(-\kappa T + \kappa t))\varphi_S \tag{3.10}$$

with the terminate boundary condition

$$\varphi(S, N, T) = 0 \tag{3.11}$$

To simplify the analysis without losing generality, let $Z = S - \bar{S}$, then it becomes

$$0 = \varphi_t + \frac{1}{2}\sigma_S^2(\varphi_Z^2 + \varphi_{ZZ}) + (-\kappa Z - \gamma N \sigma_S^2 \exp(-\kappa T + \kappa t))\varphi_Z \tag{3.12}$$

The characteristic curve for (3.12) related linear homogeneous equations is

$$dZ/dt = (-\kappa Z - \gamma N \sigma_S^2 \exp(-\kappa T + \kappa t))$$

By solving this equation, now we can set $\xi = (Z + \frac{1}{2\kappa}\gamma N \sigma_S^2 \exp(-\kappa T + \kappa t)) \exp(\kappa t)$ and let $\varphi = U(\xi)$, then (3.12) becomes

$$(\frac{dU}{d\xi})^2 + \frac{d^2U}{d\xi^2} = 0$$

with general solutions as

$$U(\xi) = \ln(\xi + C_1) + C_2 \tag{3.13}$$

where C_1 and C_2 depends on N only for there are only partial derivatives over S and t in (3.12). Therefore,

$$\varphi = \ln\left(\left(S - \bar{S} + \frac{1}{2\kappa}\gamma N \sigma_S^2 \exp\left(-\kappa T + \kappa t\right)\right) \exp\left(\kappa t\right) + C_1\right) + C_2 \tag{3.14}$$

However, the general solution can not satisfy the terminate boundary condition (3.11), which means the only solution for φ is the trivial solution $\varphi = 0$. Therefore, (3.6) is the unique solution for (3.5) under the terminate boundary condition (3.2).

CHAPTER 4: Optimal Trading in Finite Time with Fixed and Proportional Fees

4.1 Basic model

In the cases with both transaction fees, the fixed transaction fee F is not zero and we set it to 5.0 in our numerical analysis. Following [7], we conjecture that there are six optimal trading boundaries. Besides the optimal buying price $\underline{S}$ and the optimal selling price $\overline{S}$ as in the case with proportional fees in section 3, there are $\underline{N}^*$, which is the optimal number of shares in position after buying, and $\overline{N}^*$, which is the optimal number of shares in position after selling, $\underline{S}^*$, which is the optimal price after buying to reach N shares, and $\overline{S}^*$, which is the optimal price after selling to reach N shares. When the investors are in the BT, they have to buy $\underline{N}^* - N$ to reach the optimal number of shares after buying. When the investors are in the ST, they have to sell $N - \overline{N}^*$ to reach the optimal number of shares after selling. On the boundary of optimal buying price $\underline{S}$, the transaction satisfies $V(X, \underline{S}, N, t) = V(X - F - \delta\underline{S}(\underline{N}^* - N), \underline{S}, \underline{N}^*, t)$, by inserting (3.1), we have

$$g(\underline{S}, N, t) = \gamma[F + \delta\underline{S}(\underline{N}^* - N)] + g(\underline{S}, \underline{N}^*, t) \tag{4.1}$$

As $\underline{S}^*$ is the optimal price after buying to reach N shares, the value function will not change by small change of number of shares, it satisfies $V(X - \delta\underline{S}^* d0 - F, \underline{S}^*, N, t) = V(X - \delta\underline{S}^* dI - F, \underline{S}^*, N + dI, t)$. By inserting (3.1), we have

$$-\gamma\delta\underline{S}^* = g_N(\underline{S}^*, N, t) \tag{4.2}$$

Following the same way in derivation of (4.1) and (4.2), we have the other two optimal boundary conditions for $\overline{N}^*$

$$g(\overline{S}, N, t) = \gamma[F + \alpha\overline{S}(N - \overline{N}^*)] + g(\overline{S}, \overline{N}^*, t) \tag{4.3}$$

and for $\overline{S}^*$

$$\gamma\alpha\overline{S}^* = g_N(\overline{S}^*, N, t). \tag{4.4}$$

The terminate boundary condition (3.2) conflicts with the above optimal spatial boundary condition, thus we need to improve the terminate boundary condition according to the mean-reverting feature of the security price. Also we have to notice that the terminate boundary condition is critical for solving the optimal trading problem based on the equation (3.5).

As having the mean-reverting character, the security price S is approaching to the mean value $\bar{S}$ as t approaching T. And moreover, according to the trading experience and also the analytical result for the case with only proportional fees, the trading activity will stop before time boundary T. Thus from the point of view of dynamic systems, there might be bifurcations or singularities at the time close to T and $g(S, N, t)$ may not be bounded all the time from 0 to T. If we observe this dynamic system backwards at time T, not time t=0, we will see the domain of security price $S = S(t)$ is mostly divergent from the mean price $\bar{S}$ at t=T and varies along the time. Then it is feasible to consider a temporal boundary condition $g(\bar{S}, N, T) = 0$, $g_N(\bar{S}, N, T) = 0$ instead of $g(S, N, T) = 0$ as a result of the mean-reverting character of security price process. Although the temporal boundary condition $g(\bar{S}, N, T) = 0$, $g_N(\bar{S}, N, T) = 0$ is weaker than $g(S, N, T) = 0$ for all S and N, it is more practical and more feasible to be handled. Due to mean-reverting feature, the large κT implies that the security price is close to its mean value. Therefore κT needs to be large in numerical simulations and we will show this point in the numerical result. Now we have the improved terminate boundary conditions for cases with both proportional and fixed transaction fees as

$$g(\bar{S}, N, T) = 0, \tag{4.5}$$

$$g_N(\bar{S}, N, T) = 0, \tag{4.6}$$

where $\bar{S}$ is the mean security price. After adjustment, it is feasible that the equation (3.5) with all the spatial boundary conditions (3.3, 3.4, 4.1, 4.2, 4.3, 4.4) and the terminate boundary conditions (4.5, 4.6) can be satisfied simultaneously and moreover the improved temporal boundary conditions are inspired by the mean-reverting character of security price.

4.2 Analytical solutions

Following the discussion in the section 3.3, the solutions for the optimal trading with fixed transaction fee and proportional transaction fee is in the pattern as the special solution in (3.6) add the general solutions in (3.14) with two constraints $C_1(N)$ and $C_2(N)$, which depend on N only as derived in the section 3.3.

$$\begin{aligned} g(S, N, t) =& \frac{\gamma^2 N^2 \sigma_S^2}{4\kappa}(1 - \exp(-2\kappa T + 2\kappa t)) + \gamma N(S - \bar{S})(1 - \exp(-\kappa T + \kappa t)) \\ &+ \ln((S - \bar{S} + \frac{1}{2\kappa}\gamma N \sigma_S^2 \exp(-\kappa T + \kappa t)) \exp(\kappa t) + C_1) + C_2 \end{aligned} \tag{4.7}$$

Applying (4.7) to all the boundary conditions, we have first of all, at the finite time T, (4.5) becomes

$$g(\bar{S}, N, T) = \ln(\frac{1}{2\kappa}\gamma N \sigma_S^2 \exp(\kappa T) + C_1(N)) + C_2(N) = 0 \tag{4.8}$$

Differentiating to N on both sides of (4.7), (4.6) becomes

$$g_N(\bar{S}, N, T) = \frac{\frac{1}{2\kappa}\gamma \sigma_S^2 \exp(\kappa T) + C_1'(N)}{\frac{1}{2\kappa}\gamma N \sigma_S^2 \exp(\kappa T) + C_1(N)} + C_2'(N) = 0 \tag{4.9}$$

The general solution for (4.8) (4.9) is $C_2(N) = C_2$, a constant, and $C_1(N) = \exp(-C_2) - \frac{\gamma N\sigma_S^2 \exp(\kappa T)}{2\kappa}$. Then the analytical solution (4.7) becomes

$$\begin{aligned} g(S,N,t) =& \frac{\gamma^2 N^2 \sigma_S^2}{4\kappa}(1-\exp(-2\kappa T+2\kappa t)) + \gamma N(S-\bar{S})(1-\exp(-\kappa T+\kappa t)) \\ &+ \ln((S-\bar{S}+\frac{\gamma N\sigma_S^2}{2\kappa}\exp(-\kappa T+\kappa t))\exp(\kappa t)+\exp(-C_2)-\frac{\gamma N\sigma_S^2\exp(\kappa T)}{2\kappa})+C_2 \end{aligned} \tag{4.10}$$

To better understand what C_2 stands for, we need to differentiate both sides of (4.10) over S and we have

$$\begin{aligned} g_S(S,N,t) =& \gamma N(1-\exp(-\kappa T+\kappa t)) \\ &+ \frac{\exp(\kappa t)}{(S-\bar{S}+\frac{\gamma N\sigma_S^2}{2\kappa}\exp(-\kappa T+\kappa t))\exp(\kappa t)+\exp(-C_2)-\frac{\gamma N\sigma_S^2\exp(\kappa T)}{2\kappa}} \end{aligned}$$

Then $g_S(\bar{S},N,T) = \exp(C_2+\kappa T)$, thus

$$C_2 = -\kappa T + \ln(g_S(\bar{S},N,T)) \tag{4.11}$$

By mean reverting features, the security price fluctuates around the mean value as t approaches terminate time T. The extreme case is that $S = \bar{S}$ only at terminate time T, which implies g_S approach to infinity as t approaches to T at $S = \bar{S}$. Thus in the numerical simulations, C_2 will be set to large values and numerical solutions with different C_2 will be presented. After differentiating (4.10) over N, we have

$$
\begin{aligned}
g_N(S,N,t) =& \frac{\gamma^2 N\sigma_S^2}{2\kappa}(1-\exp(-2\kappa T+2\kappa t)) + \gamma(S-\bar{S})(1-\exp(-\kappa T+\kappa t)) \\
&+ \frac{\frac{\gamma\sigma_S^2}{2\kappa}(\exp(-\kappa T+\kappa t)-\exp(\kappa T-\kappa t))}{S-\bar{S}+\frac{\gamma N\sigma_S^2}{2\kappa}\exp(-\kappa T+\kappa t)+\exp(-C_2-\kappa t)-\frac{\gamma N\sigma_S^2\exp(\kappa T-\kappa t)}{2\kappa}} \\
=& B_1 S + B_0 + \frac{D_2}{S+D_0}
\end{aligned}
\tag{4.12}
$$

Then $g_N = AS$ ($A=-\gamma\delta$ for (3.3) and (4.2), $A=\gamma\alpha$ for (3.4) and (4.4)) becomes a quadratical equation of S

$$
0 = (B_1-A)S^2 + (B_1D_0+B_0-AD_0)S + (B_0D_0+D_2) \tag{4.13}
$$

and the determinants Δ is

$$
\Delta = (B_1D_0+B_0-AD_0)^2 - 4(B_1-A)(B_0D_0+D_2) \tag{4.14}
$$

where

$$
\begin{aligned}
B_1 &= \gamma(1-\exp(-\kappa T+\kappa t)), \\
B_0 &= \frac{\gamma^2 N\sigma_S^2}{2\kappa}(1-\exp(-2\kappa T+2\kappa t)) - \gamma\bar{S}(1-\exp(-\kappa T+\kappa t)), \\
D_2 &= \frac{\gamma\sigma_S^2}{2\kappa}(\exp(-\kappa T+\kappa t)-\exp(\kappa T-\kappa t)), \\
D_0 = -\bar{S} &+ \frac{\gamma N\sigma_S^2}{2\kappa}\exp(-\kappa T+\kappa t)+\exp(-C_2-\kappa t)-\frac{\gamma N\sigma_S^2\exp(\kappa T-\kappa t)}{2\kappa}.
\end{aligned}
$$

As the same as in cases with proportional fees only, once $t = t_c = T + \frac{\ln(1-\alpha)}{\kappa}$, singularity may happen in (4.13) as converting from quadratic equation to linear equation. And numerical simulation will limit between $t=0$ and $t=t_c-dt_c$, where dt_c is very small comparing to T. For $t\in[0,t_c)$, $(B_1+\delta) > (B_1-\alpha) > 0$, then we can do theoretical analysis on (18) under $(B_1-A) > 0$. And moreover $D_2 \le D_2(t_c) < 0$ for it is an increasing function of t. Thereafter, let $q = (B_1 - A) > 0$ for $t \in [0, t_c)$, the determinant in (4.14) can be estimated by

$$
\begin{aligned}
\Delta =& q^2 D_0^2 + B_0^2 - 2qD_0B_0 - 4qD_2 \\
=& q((qD_0^2 + \frac{B_0^2}{q}) - 2D_0B_0 - 4D_2), \\
\geq& q(2 \mid D_0B_0 \mid -2D_0B_0 - 4D_2) \\
\geq& q(0 - 4D_2) \\
>& 0,
\end{aligned}
$$

which indicates that there are always two different solution for $g_N = \gamma\alpha S$ and $g_N = -\gamma\delta S$. Therefore the optimal boundary condition (3.3,4.2) and (3.4, and 4.4) can always be satisfied for $t \in [0, t_c)$.

After solving optimal trading boundaries $\underline{S}$, $\underline{S}^*$, $\overline{S}$ and $\overline{S}^*$, the optimal positions $\underline{N}^*, \overline{N}^*$ need to be solved from (4.1) and (4.3), which include the fixed transaction fee. As $\underline{N}^* = \underline{N}^*(\underline{S})$, and $\underline{S} = \underline{S}(N, t)$, $\underline{S}$ and $\underline{N}^*$ are dynamic processes depending on N and t, while these optimal trading boundaries do not depend on time t in the optimal trading problem for infinite time. Thus in the numerical result, we will present the numerical solutions of $\underline{N}^*$ and $\overline{N}^*$ for different N and t.

4.3 Optimal trading strategy by numerical analysis

In this section, the parameters in numerical analysis are set to $\gamma = 0.001, \sigma_S = 0.5, \bar{S} = 40, \kappa = 0.1, N = 100, \alpha = \delta = 0.0025, F = 5.0, T = 20, C_2 = 10$ in most numerical samples. It is mentioned in section 4.2 that the constant C_2 in (4.10) needs to be large in order to satisfy the mean-reverting feature for security price, as the probability density of security price is huge around $\bar{S}$ at time T. This point is confirmed by the numerical analysis result in Fig 4.1 with $C_2 = -10, -5, -1, 10$ and Fig 4.2 with $C_2 = -1, 10, 1000$. In Fig 4.1, the optimal trading boundary

$\underline{S}, \overline{S}^*$ are negative for $C_2 = -10, -5$, while they are just a bit less than $\bar{S}$ for $C_2 = -1, 10$. In Fig 4.2, the difference between optimal trading prices with $C_2 = 10$ and optimal trading prices with $C_2 = 1000$ is at most $10^{-4.6}$ and the difference decreases with time. It shows the optimal trading prices for different C_2 approach fast to the extreme case with $C_2 \to \infty$, which is the extreme mean-reverting case that the security price can only equal to the mean value at the terminate time T. As investors use at most the last two decimal digits for security prices, we set C_2 to be 10 in the following testing cases and the difference to the extreme mean reverting case is less than one percent cent as in Fig 4.2.

At different time points, the numerical solution for optimal trading prices are obtained by applying (4.12) to (3.3,3.4,4.2,4.4) as in Fig 4.3. The curves of optimal buying price in the first figure of Fig 4.3 are concave down, unlike the straight forward optimal buying price curves in the first figure of Fig 3.2. It indicates that for small amount of shares, the fixed transaction fee really matters, however, for large amount of shares, the influence of fixed transaction fee in the optimal trading strategy reduces. For small N, the fixed transaction fee are more than or close to the proportional transaction fee. Then the larger optimal buying price is required for earlier time t in the first figure of Fig 4.3, which indicates that the uncertainty at t=10 and 15 is not as much as it is at t=5, for the assumption of mean-reverting, which ensures that the security price will go to the mean value at terminate time T=20. As shown in the last figure of Fig 4.3, The optimal selling price $\overline{S}(N, t)$ is close to the optimal price after buying $\underline{S}^*$, and the optimal buying price $\underline{S}(N, t)$ is close to the optimal price after selling $\overline{S}^*$.

The curves of optimal trading price for different time horizons are presented in Fig 4.4 for T = 20, 40 and 60. It shows that the optimal trading activity depends on how much time is left to the terminate time T. For example, The optimal trading activity at t=0 for T=20 is close to the optimal trading activity at t=40 for T=60. The optimal trading strategies suggest the investors

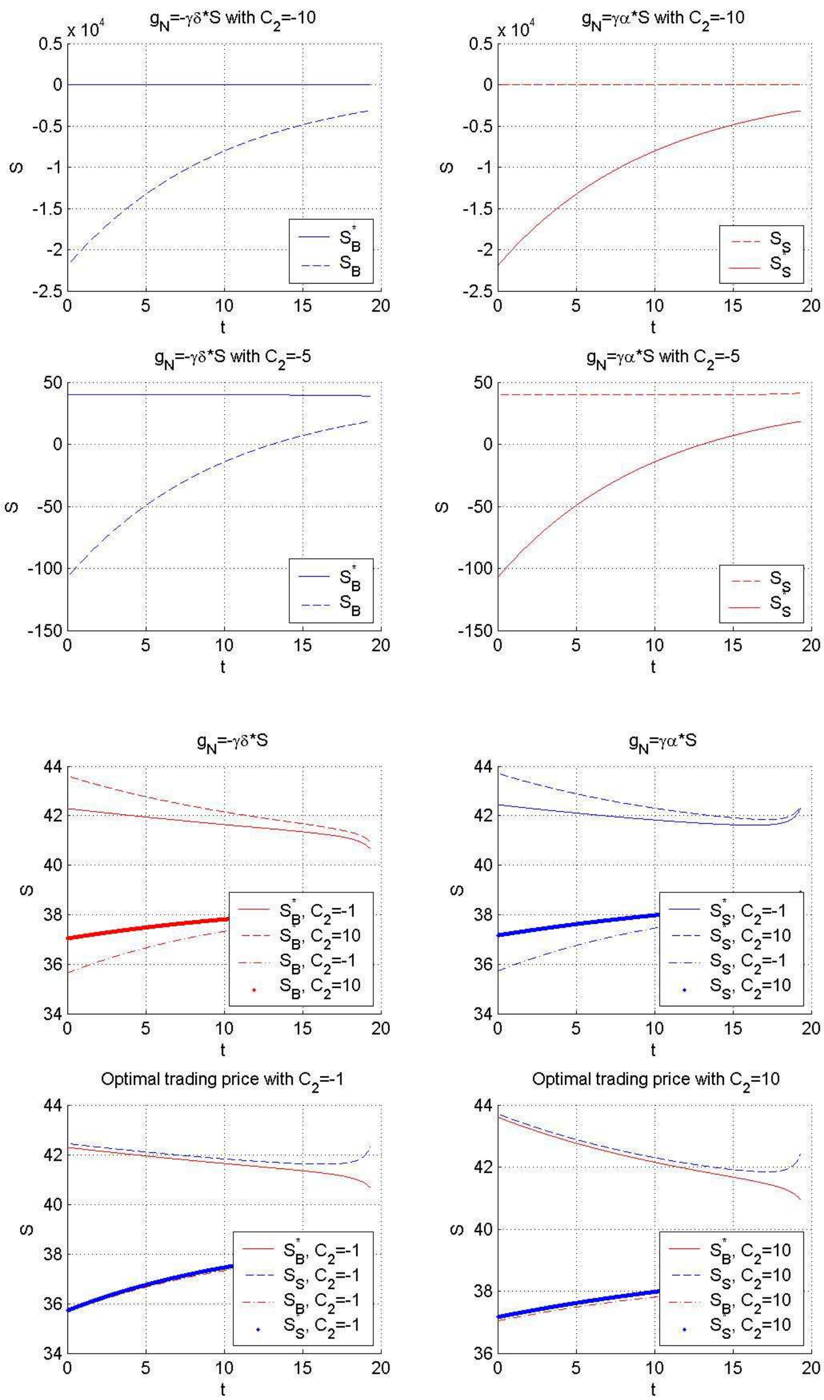
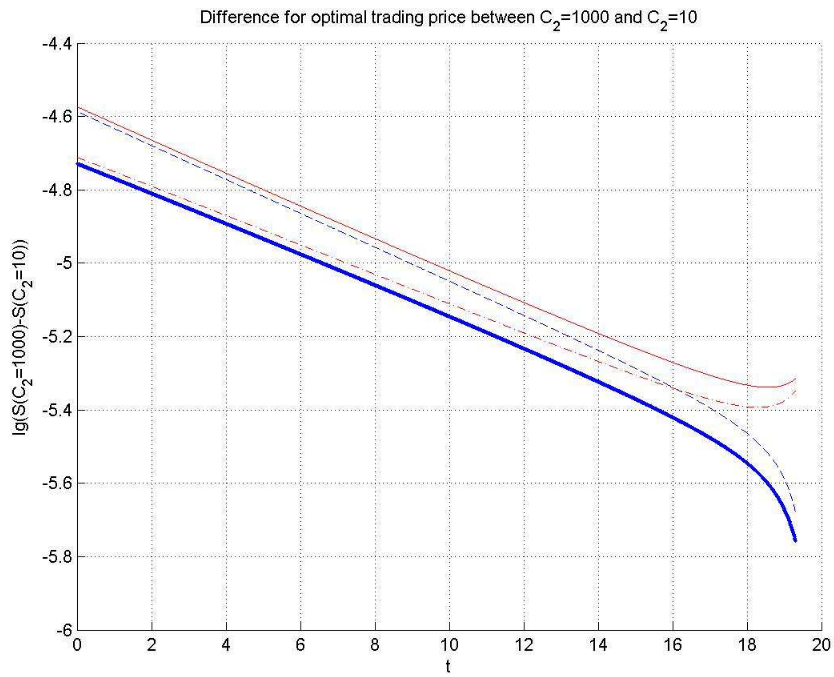


Figure 4.1: Optimal trading prices with different constants C_2 in the cases with both fixed and proportional fees in finite time



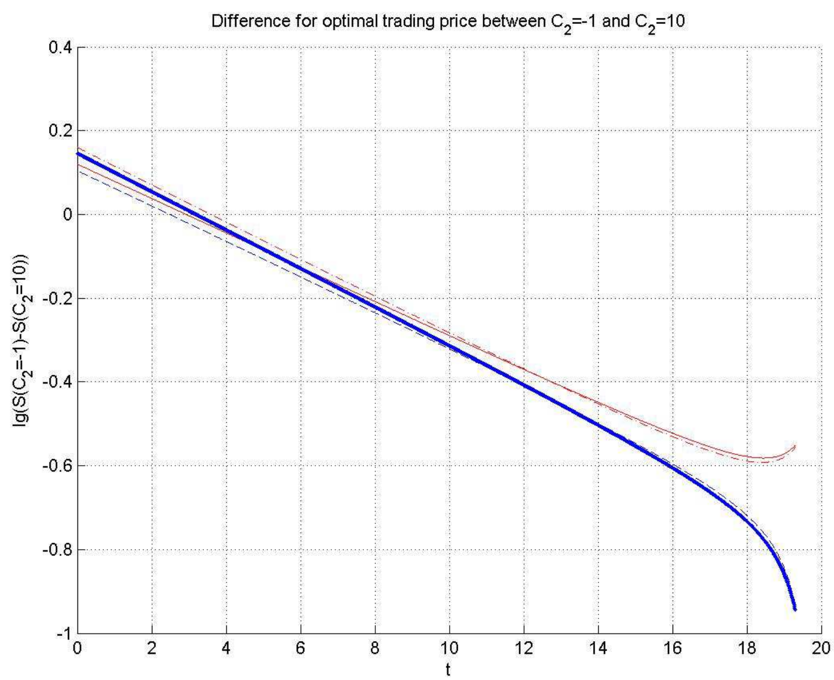


Figure 4.2: Difference in optimal trading prices by using different constants in the cases with both fixed and proportional fees in finite time

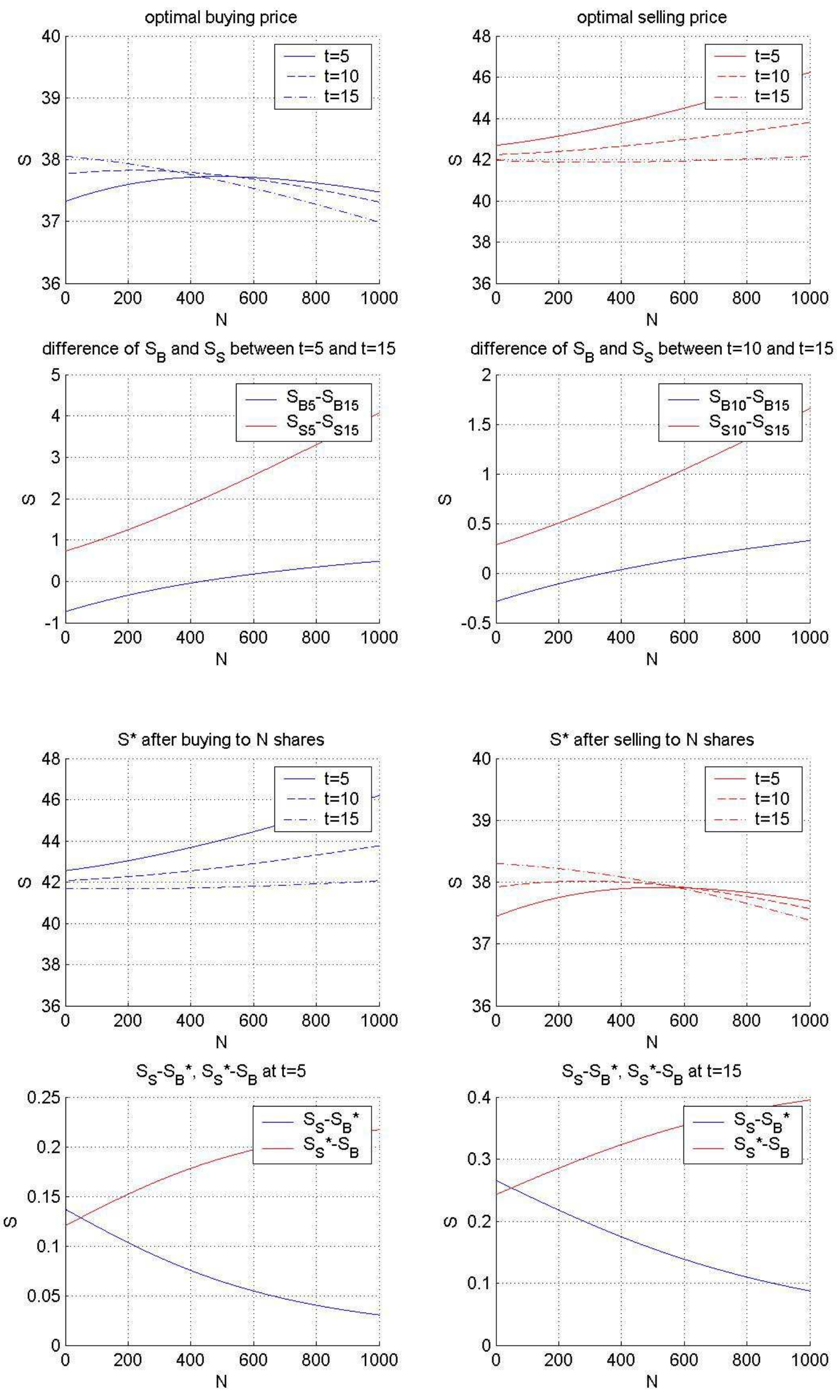


Figure 4.3: Optimal trading prices at different time points in the cases with both fixed and proportional fees in finite time

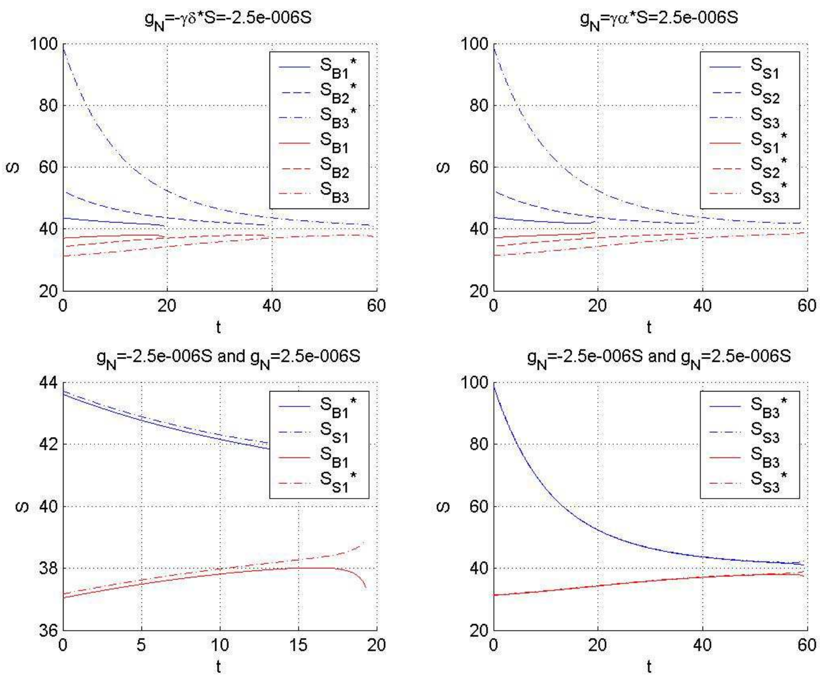


Figure 4.4: Optimal trading prices with different time ranges in the cases with both fixed and proportional fees in finite time

with longer time horizon to buy at lower price and sell at higher price than the investors with shorter time horizon do. Also at the optimal selling price changes more than the optimal buying price along with time, which coincides with our common sense in real trading.

Once increasing the strength of mean-reverting, the absolute value of the rate of change for optimal trading price along the time will increase also, for the mean-reverting strength κ indicates how much time will it cost the security price to go back to the mean value as in Fig 4.5 with $\kappa = 0.01, 0.1, 0.2$. For the extreme small mean-reverting strength as $\kappa = 0.01$, the optimal trading price is really flat for the time far from the terminate time T, while the optimal trading price varies fast along the time for $\kappa = 0.2$. The intra-day traders need to pick up securities with κ not too large, otherwise, their target price has to modify in very short time. However, if the mean-reverting strength is too small, it would cost too much time for the price to return to the mean value and the intra-day traders need to be more patient.

For the optimal trading price with different proportional transaction fees and the optimal trading price with different volatility, the comparison results are similar as the results in the cases with proportional fees only. And moreover, the fixed transaction fees only influence the optimal positions $\underline{N}^*$ and $\overline{N}^*$. As shown in Fig 4.6 that the curves for optimal position with $F = 10$ is always above those for $F = 5$, increasing fixed transaction fee F leads to selling more or buying more shares when trading happens. In other words, more fixed transaction fee F, more trading quantity is required to reach the optimal position in trading. Without fixed cost, the optimal positions $\underline{N}^*$ and $\overline{N}^*$ are equal to N as the curve for F=0 in Fig 4.6, for the investors are on the boundary of NT with $\overline{S}$ or $\underline{S}$ at this time. With fixed transaction fee, the investors need to trade to deduct the influence of fixed cost as the curves for F=5 and F=10 in Fig 4.6. For buying up to $\underline{N}^*$ from N, the investors with more amount of shares N have to buy more. For instance, as the two figures in the upper level of Fig 4.6, at the initial time, the investors with N=200 need to buy about 50 shares while the investors with N=100 need to buy about 36 shares. However, it is opposite for investors selling up to $\overline{N}^*$ as in the lower level of Fig 4.6, the investors with N=200 just need to sell about 12 shares, while the investors with N=100 need to sell about 21 shares. One reasonable explanation may be that when selling to the optimal position $\overline{N}^*$, the profit in account for holding 200 shares is more than it for holding 100 shares as the security price at that time is above mean value, thus the investors with 200 shares in hand no need to sell more than investors with 100 shares in hand. When buying to the optimal position $\underline{N}^*$, the loss in account for holding 200 shares is more than it for holding 100 shares as the security price at that time is below mean value, thus the investors with 200 shares in hand need to buy more than investors with 100 shares in hand.

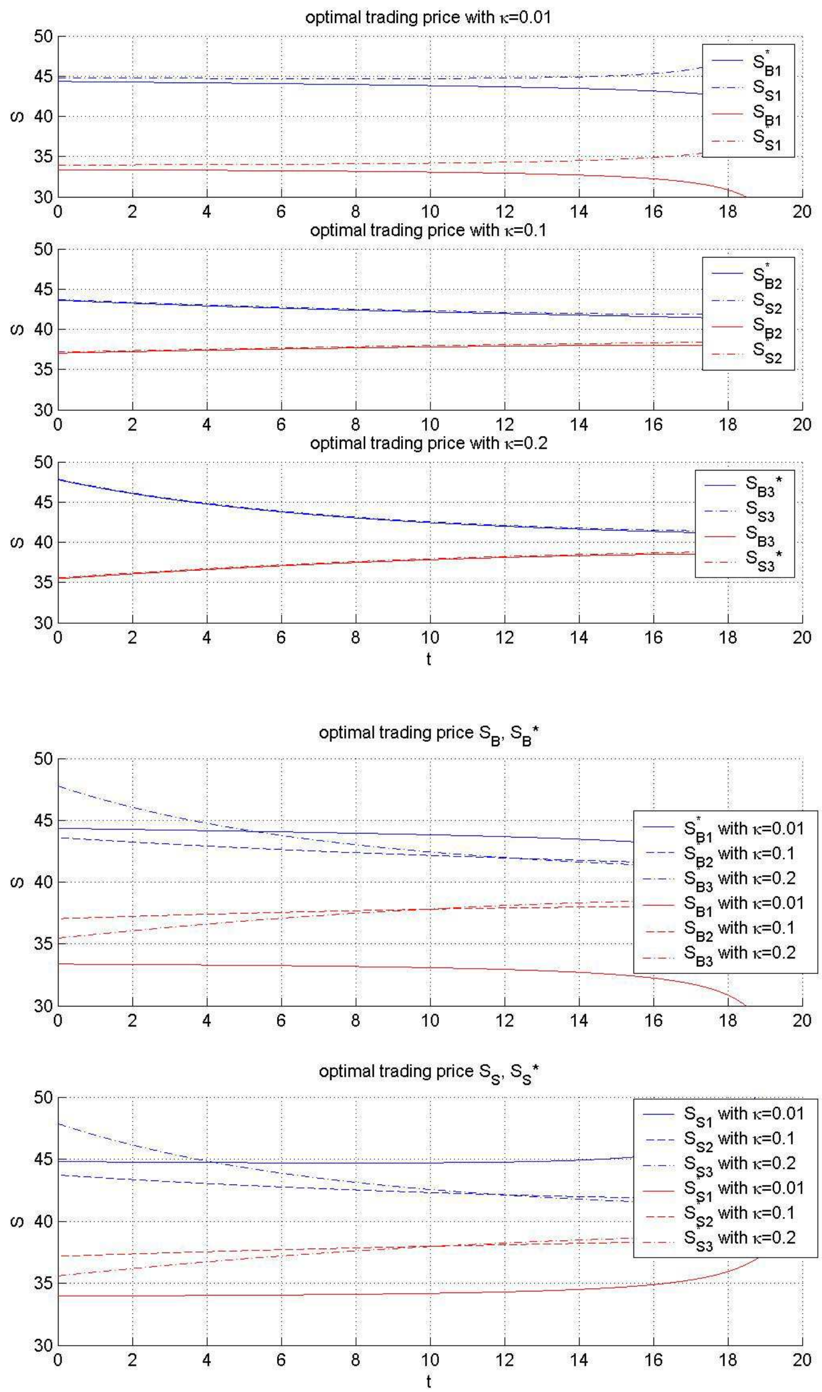


Figure 4.5: Optimal trading prices with different mean-reverting rates in the cases with both fixed and proportional fees in finite time

As mentioned in the section 4.2, the optimal positions $\underline{N}^*$ and $\overline{N}^*$ depend on N and t. They are solved from (4.1) and (4.3) numerically. For instance, if there is no testing value of $\underline{N}^*$ to satisfy (4.1), we choose the one minimizing the difference between the left hand side and the right hand side of (4.1) and record the error equal to the absolute value of the minimum difference. There are numerical results for the optimal position with different positions N at certain time in Fig 4.7.

There is a sharp turning in the curve of optimal buying position at N between 800 and 900 in Fig 4.7. And we may explain this sharp turning by showing the errors (RHS-LHS of (4.1)) in Fig 4.8. For large N (1000 is large comparing to 100), at t=10, the error at $\underline{N}^* = N$ is already very close to zero, and however, the curve of error in Fig 4.8 does not cross $Error = 0$. Thus we pick up $\underline{N}^*$ with minimum error about 2×10^{-3}.

And moreover as in Fig 4.9, at time t=15 the solution for optimal buying $\underline{N}^*$ exists, in contrast to the situation at time t=0 and t=10. The right figure in Fig 4.9 shows that there are two solution of (4.1) for $\underline{N}^*$ and we pick up the solution more close to N, for once the investors trade enough amount to reach an optimal position, they would not trade any more shares. And moreover, there is no need to present the errors for optimal selling positions $\overline{N}^*$, for they always exist in our numerical simulations as the left lower figure in Fig 4.9.

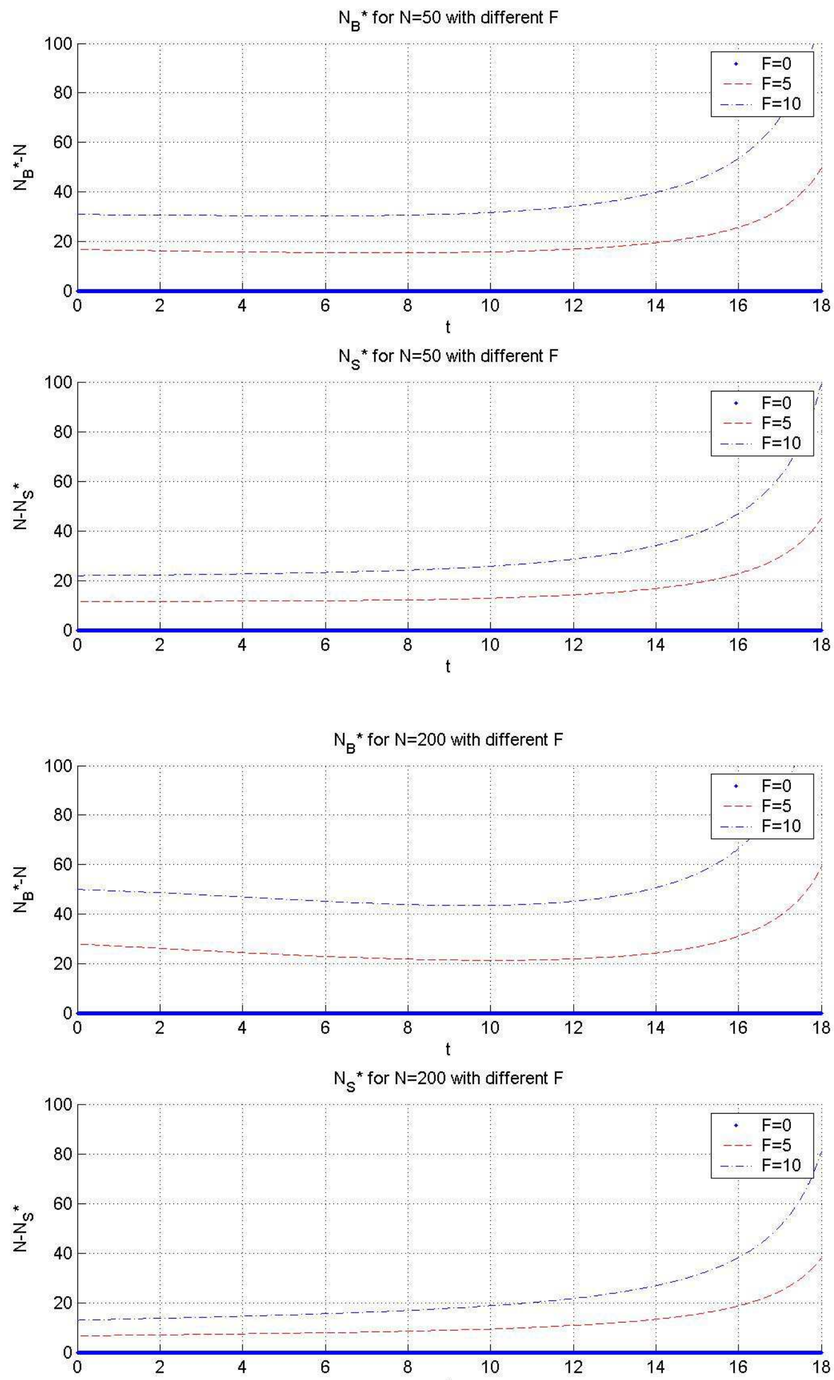


Figure 4.6: Optimal trading positions for different fixed transaction fees in the cases with both fixed and proportional fees in finite time

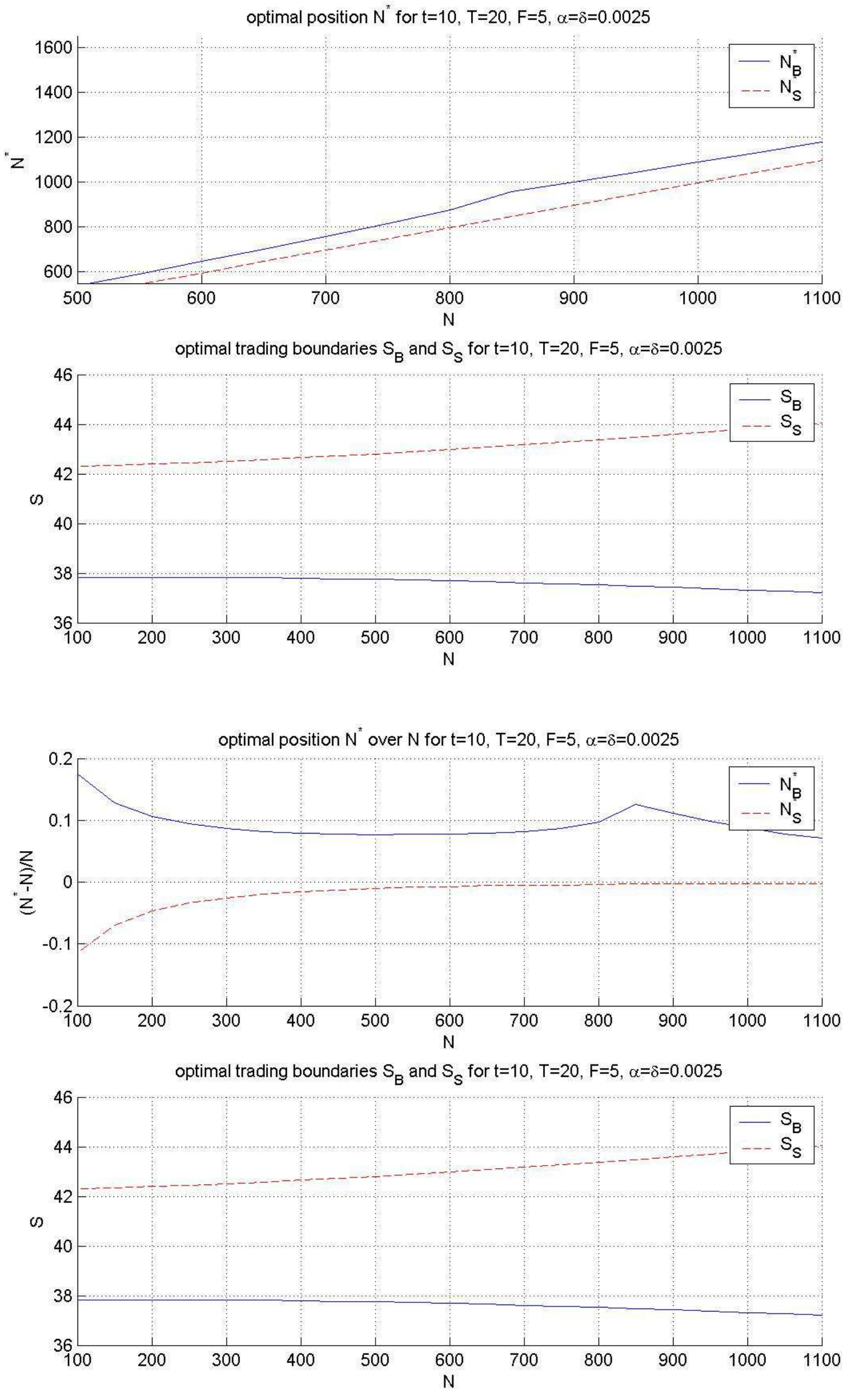


Figure 4.7: Optimal trading positions for different number of shares in the cases with both fixed and proportional fees in finite time

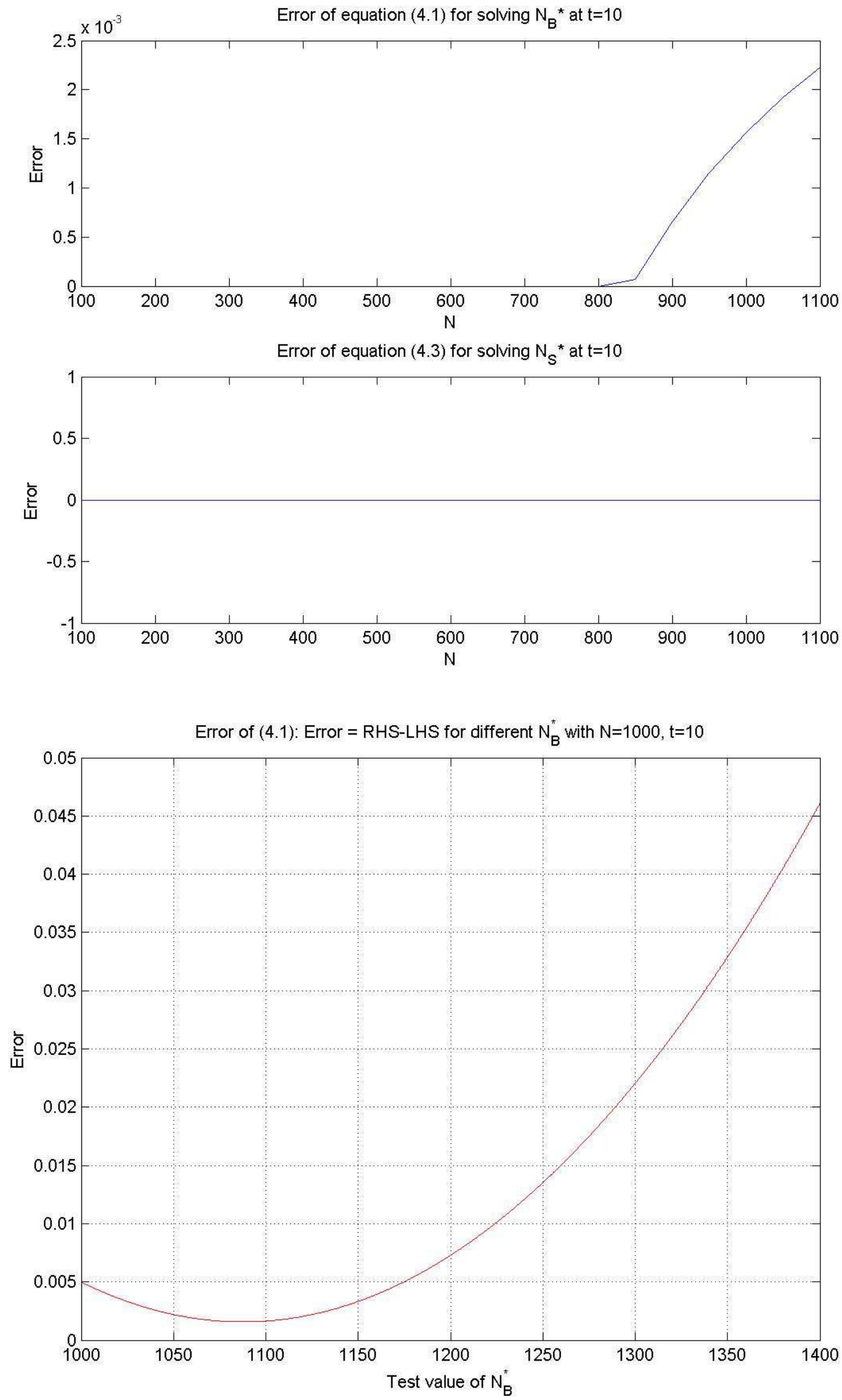
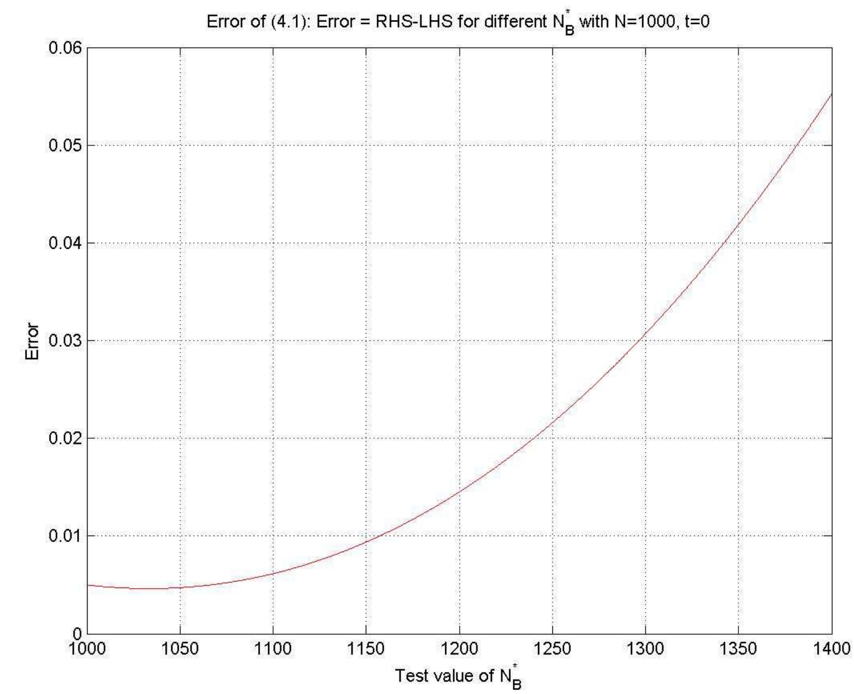


Figure 4.8: Errors (RHS-LHS for (4.1)) in numerical resolution for optimal trading positions



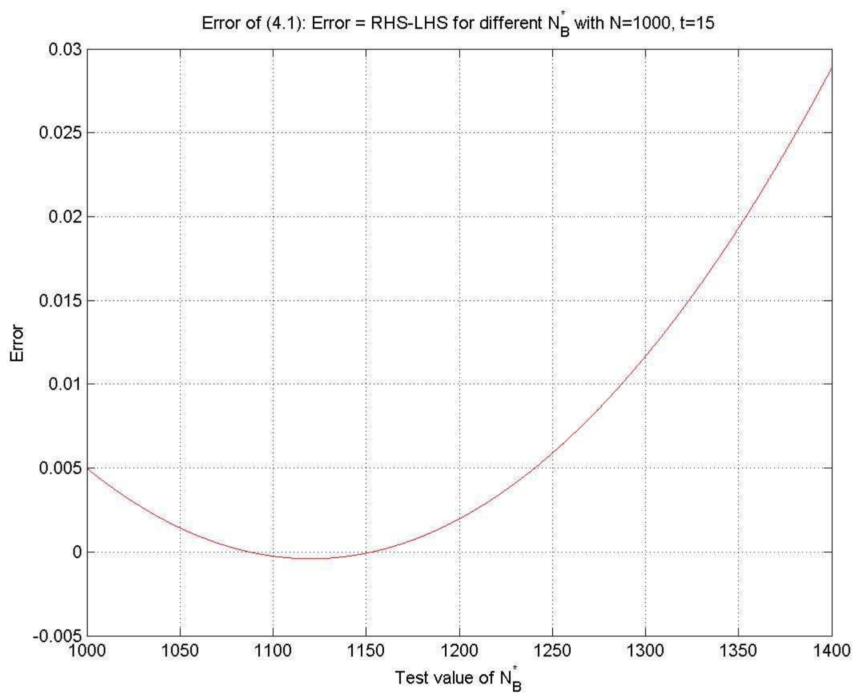


Figure 4.9: Errors (RHS-LHS for (4.1)) in numerical resolution at different time points for optimal trading positions

CHAPTER 5: Conclusion

5.1 Conclusion

This thesis studies the optimal trading problem in finite time, including cases without transaction fees in chapter 2, cases with only proportional transaction fees in chapter 3 and also cases with both fixed transaction fee and proportional transaction fees in chapter 4. For these three cases, basic models are built, analytical solutions are presented and verified, and optimal trading boundaries are solved numerically. There are several points in theoretical analysis need to be mentioned in the conclusion part.

First of all, as the security price and number of shares are considered separately in this thesis, unlike in [7, 11], the partial differential equations in this thesis are complicated. And the mean-reverting feature is taken advantage to comprehend and solve the equations having features of travelling waves. And we solve the equations by emphasizing the terminal boundary at t=T, unlike emphasizing the spatial boundary as in [7, 11].

Secondly, for the cases with fixed and proportional transaction fees, by the help of (4.1), (4.3)(the optimal boundary condition (3.3),(3.4) can be derived from these two), we can write out $g(S,N,t)$ for the trading region in the same pattern and just need to replace $\underline{S}$ and $\overline{S}$ by S in (4.3) and (4.4). While for the cases with the proportional transaction fees only, the solution for the trading region is not unique so far, for there are only (3.3), (3.4) to describe the g_N. And general solutions of $g(S,N,t)$ in the trading region can be built by the continuities of $g(S,N,t)$ and $g_N(S,N,t)$ at the optimal trading boundaries $\underline{S}$ and $\overline{S}$. For instance, the general pattern for the buying region is as $g(S, N, t) = \phi(S, t) + g(\underline{S}, N, t)$, and $\phi(S, t)$ only needs to satisfy $\phi(\underline{S}, t) = 0$. Therefore, up to now, the specific functions for the trading region in the cases with proportional fees only are still not clear.

Thirdly, by the theoretical analysis of partial differential equations, it seems like the space of general solutions shrinks to a single point as the result of the terminal boundary condition $g(S, N, T) = 0$ and the mean-reverting features in the security price.

Finally, the idea to generate the basic model for the cases with both fixed and proportional transaction fees, especially the terminate boundary conditions (4.5) and (4.6), is for the existence of two solutions for one equation, as $\underline{S}$ and $\underline{S}^*$ are the solutions for $g_N = -\gamma\delta S$, $\overline{S}$ and $\overline{S}^*$ are the solutions for $g_N = \gamma\alpha S$. Therefore, in this thesis we artificially build the terminate boundary conditions as (4.5) and (4.6). And fortunately, these terminate boundary conditions reflect the mean-reverting feature very well.

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